

# A proof of the butterfly theorem using the scale factor between the two wings.

Martin Celli

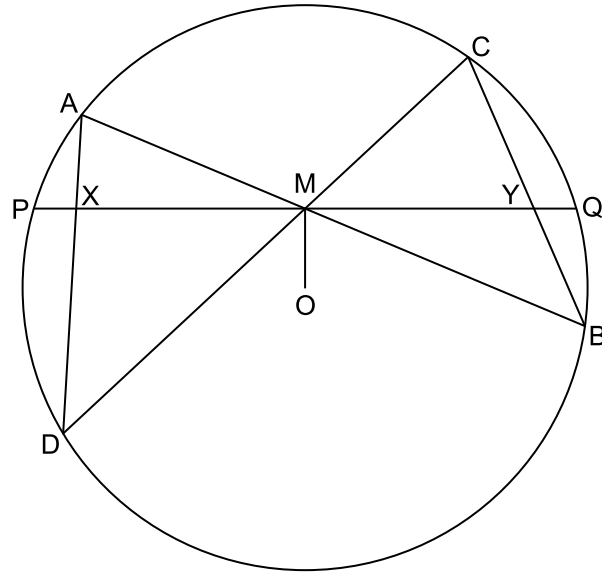
October 21st 2016

Departamento de Matemáticas  
 Universidad Autónoma Metropolitana-Iztapalapa  
 Av. San Rafael Atlixco, 186. Col. Vicentina. Del. Iztapalapa. CP 09340. México, D.F.  
 E-mail: cell@xanum.uam.mx

**Abstract.** We give a new proof of the butterfly theorem, based on the use of several expressions involving the scale factor between the two wings.

The aim of this article is to propose a new proof of the following theorem:

**Butterfly theorem.** *Let  $M$  be the midpoint of a chord  $PQ$  of a circle, through which two other chords  $AB$  and  $CD$  are drawn. Let us assume that  $A$  and  $D$  do not belong to a same half-plane defined by  $PQ$ . Let  $X$  (respectively  $Y$ ) be the intersection of  $AD$  (respectively  $BC$ ) and  $PQ$ . Then  $M$  is also the midpoint of  $XY$ .*



Let  $O$  be the center of the circle. The points  $A$  and  $C$  belong to a same half-plane defined by  $PQ$ , the points  $B$  and  $D$  to the other half-plane. For sake of simplicity, we can assume that  $O$  belongs to the same half-plane as  $B$  and  $D$ .

Several classic and recent proofs of this theorem are known ([1], [2]). In our proof, we directly show that the ratio  $\sin(CYM)/\sin(AXM)$  is nothing but the scale factor between the two wings  $AMD$  and  $CMB$ , which are similar by the inscribed angle theorem:

$$\frac{\sin(CYM)}{\sin(AXM)} = \alpha, \text{ where } \alpha = \frac{CM}{AM} = \frac{BM}{DM} = \frac{CB}{AD}.$$

More precisely, we have:

$$\sin(AXM) = \frac{DM^2 - AM^2}{AD \cdot OM}.$$

As a matter of fact:

$$\begin{aligned} DM^2 - AM^2 &= ||\overrightarrow{DO} + \overrightarrow{OM}||^2 - ||\overrightarrow{AO} + \overrightarrow{OM}||^2 \\ &= OD^2 + OM^2 + 2\overrightarrow{DO} \cdot \overrightarrow{OM} - (OA^2 + OM^2 + 2\overrightarrow{AO} \cdot \overrightarrow{OM}) \\ &= 2\overrightarrow{DO} \cdot \overrightarrow{OM} \text{ as } OA = OD \\ &= 2AD \cdot OM \sin(AXM) \text{ as } OMX = OMY = \frac{OMX + OMY}{2} = \frac{\pi}{2}, \end{aligned}$$

because triangles  $OMP$  and  $OMQ$  are congruent. Similarly, we have:

$$\sin(CYM) = \frac{BM^2 - CM^2}{CB \cdot OM}.$$

Thus:

$$\frac{XM}{YM} = \frac{AM}{CM} \times \frac{CM}{YM} \times \frac{XM}{AM} = \frac{AM}{CM} \times \frac{\sin(CYM)}{\sin(YCM)} \times \frac{\sin(XAM)}{\sin(AXM)}$$

by the law of sines, applied to triangles  $AXM$  and  $CYM$

$$\begin{aligned} &= \frac{AM}{CM} \times \frac{\sin(CYM)}{\sin(AXM)} \text{ by the inscribed angle theorem} \\ &= \frac{AM}{CM} \times \frac{BM^2 - CM^2}{CB \cdot OM} \times \frac{AD \cdot OM}{DM^2 - AM^2} = \frac{AM}{CM} \times \frac{AD}{CB} \times \frac{BM^2 - CM^2}{DM^2 - AM^2} \\ &= \frac{1}{\alpha} \times \frac{1}{\alpha} \times \alpha^2 = 1. \end{aligned}$$

## References.

[1] Alexander Bogomolny, Butterfly theorem. Interactive Mathematics Miscellany and Puzzles:

<http://www.cut-the-knot.org/pythagoras/Butterfly.shtml>

[2] Cesare Donolato, A proof of the butterfly theorem using Ceva's theorem. Forum Geometricorum, vol. 16 (2016), 185-186.

Dr. Martin Celli.

Depto. de Matemáticas, Universidad Autónoma Metropolitana-Iztapalapa. México, D.F.

E-mail: cell@xanum.uam.mx